24-Mar L13

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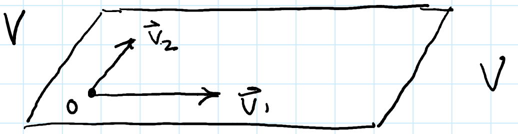
§3.3 Dimensions of a Subspace in $\mathbb{R}^n$

given plane V in $\mathbb{R}^3$

need 2 non-parallel vectors to span V

- 1 vector is not enough to represent plane
- 3 vectors would be linearly dependent



$\vec{v}_1, \vec{v}_2$ form a basis of V

vocabulary review:

subspace := subset of $\mathbb{R}^n$ s.t. contains $\vec{0}$ in $\mathbb{R}^n$
closed under linear combination

- addition
- scalar mult.

Span := set of all LC $\quad \text{span}(\vec{v}_1, \ldots, \vec{v}_m) = \{c_1\vec{v}_1 + \ldots + c_m\vec{v}_m \text{ s.t. } c_i \in \mathbb{R}\}$

linearly independent := vectors are not redundant

basis := span subspace V AND are linearly independent

↑ bases (plural)

How to generalize above scenario for $\mathbb{R}^n$?

thm: given $\vec{v}_1, \ldots, \vec{v}_p$ and $\vec{w}_1, \ldots, \vec{w}_q$ in a subspace V of $\mathbb{R}^n$

if $\vec{v}_1, \ldots, \vec{v}_p$ are linearly independent **and**
if $\vec{w}_1, \ldots, \vec{w}_q$ span V
then $q \geq p$

revisit example above: V is plane ($\mathbb{R}^2$)
vector space is $\mathbb{R}^3$

→ at most 2 linearly independent vectors: $p \leq 2$
→ at least 2 vectors to span V

$$p \leq 2 \;\downarrow\; 2 \geq p \;\downarrow$$

$$\therefore q \geq 2 \rightarrow q \geq 2 \geq p$$

$$q \geq p \;\square$$

thm: all bases of a subspace V of $\mathbb{R}^n$ has same # of vectors

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ex. given $\left.\begin{array}{l}\text{line } L \\ \text{plane } V\end{array}\right]$ in $\mathbb{R}^3$

basis of L would consist of any non-zero vectors in L

all bases of V would consist of two vectors
all bases of $\mathbb{R}^3$ " three vectors

$\downarrow$

could pick $\vec{e}_1, \vec{e}_2, \vec{e}_3$

recall: $\vec{i}, \vec{j}, \vec{k}$

given subspace V of $\mathbb{R}^n$
then <u>dimension</u> or dim(V) denotes number of vectors in a basis of V

ex. given $\underset{\substack{4\times 5 \\ \text{matrix}}}{A} = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix}$

find a basis of ker(A)

$\downarrow$

$A\vec{x} = \vec{0}$

$$\text{ex. } x_1 + \overset{r}{2x_2} + 0x_3 + \overset{s}{3x_4} - \overset{t}{4x_5} = 0$$

$$x_1 = \underbrace{-2x_2 + 0x_3 - 3x_4 + 4x_5}$$

set 3 arbitrary constants

$$\text{rref}(A) = \begin{bmatrix} \textcircled{1} & 2 & 0 & 3 & -4 \\ 0 & 0 & \textcircled{1} & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivots
rank(A) = 2

let $x_2 = r$ let $x_4 = s$ let $x_5 = t$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} -2r & -3s & +4t \\ 1r & 0 & 0 \\ 0 & 4s & -5t \\ 0 & 1s & 0 \\ 0 & 0 & 1t \end{bmatrix} \\ \begin{array}{ccc} r & \quad s & \quad t \end{array} \end{array}$$

Conclude: takes 3 vectors to form a basis of ker(A)

$$= r\underbrace{\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}}_{\vec{w}_1} + s\underbrace{\begin{bmatrix}-3\\0\\4\\1\\0\end{bmatrix}}_{\vec{w}_2} + t\underbrace{\begin{bmatrix}4\\0\\-5\\0\\1\end{bmatrix}}_{\vec{w}_3}$$

Check for linear independence in $\vec{w}_1, \vec{w}_2, \vec{w}_3$

$$\begin{array}{c} \\ x_2 \\ \\ x_4 \\ x_5 \end{array} \begin{bmatrix}-2\\\textcircled{1}\\0\\\boxed{0}\\\boxed{0}\end{bmatrix} \begin{bmatrix}-3\\\boxed{0}\\4\\\textcircled{1}\\\boxed{0}\end{bmatrix} \begin{bmatrix}4\\\boxed{0}\\-5\\\boxed{0}\\\textcircled{1}\end{bmatrix}$$

i.e. each vector has a component of 1 AND its other corresponding components (in same row) are 0

notice: correspond to free variables x_2, x_4, x_5

Conclude basis of ker(A) is: $\vec{w}_1, \vec{w}_2, \vec{w}_3$

1 2 3

$\dim(\ker A) = 3$

follow up question: find a basis of im(A)

again, utilize rref(A):

$$\begin{array}{ccccc} 1 & 2 & 0 & 3 & -4 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{a}_5 \end{array}$$

this time, look at columns <u>wo</u> leading 1

$\vec{a}_2 = 2\vec{a}_1$ $\therefore$ $\vec{a}_2$ is redundant

$\vec{a}_4 = 3\vec{a}_1 - 4\vec{a}_3$ $\therefore$ $\vec{a}_4$ is redundant

$\vec{a}_5 = -4\vec{a}_1 + 5\vec{a}_3$ $\therefore$ $\vec{a}_5$ is redundant

$\vec{a}_1, \vec{a}_3$ are non-redundant vectors

basis of im(A) is $\begin{bmatrix}1\\-1\\4\\3\end{bmatrix}, \begin{bmatrix}2\\-1\\5\\1\end{bmatrix}$

get from A not rref(A)

$\therefore \dim(\text{im} A) = 2$

note this is also rank(A)

tying concepts together:

↳ note this is also rank(A)

tying concepts together:

- basis of im(A) is set of column vectors in A where columns of rref(A) have leading 1's
- $\dim(\text{im } A) = \text{rank}(A)$
- $\dim(\ker A) = \#$ of free variables

given $n \times m$ matrix A

$$= m - \text{rank}(A)$$

$$= m - \dim(\text{im } A)$$

solve for m

$$\boxed{\dim(\ker A) + \dim(\text{im } A) = m}$$

Fundamental Theorem of Lineal Algebra:

(nullity of A) + (rank of A) = m

given $T(\vec{x}) = A\vec{x}$ $\quad \mathbb{R}^m \to \mathbb{R}^n$

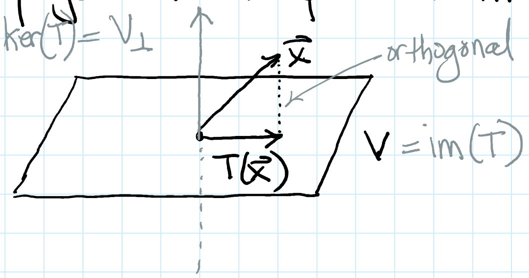
then

m is dimension of domain

nullity of A = $\dim(\ker A)$ = # dimensions that "collapse" after applying T

rank of A = $\dim(\text{im } A)$ = # dimensions that "survive" T

ex. let T = orthogonal projection onto plane V in $\mathbb{R}^3$



then
dim of domain : 3
one dim. collapses : ker T = line $V_\perp$
result is $\underset{\uparrow \, \text{im}(T)}{V}$, which is 2 dimensional

$$m - \dim(\ker T) = \dim(\text{im } T)$$
$$3 - 1 = 2$$

ex. find bases of image and kernel of:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 1 & 2 & 0 & 2 & 3 \end{bmatrix}$$

$$A = \begin{array}{c}\begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 1 & 2 & 0 & 2 & 3 \\ 1 & 2 & 0 & 3 & 4 \\ 1 & 2 & 0 & 4 & 5 \end{bmatrix} \\ \begin{matrix} \vec v_1 & \vec v_2 & \vec v_3 & \vec v_4 & \vec v_5 \end{matrix}\end{array}$$

4x5 ↓ m=5

$\vec v_2 = 2\vec v_1$

$\vec v_3 = \vec 0$ consider zero vector redundant

$\vec v_5 = \vec v_1 + \vec v_4$

∴ $\vec v_2$, $\vec v_3$, $\vec v_5$ are redundant
$\vec v_1$, $\vec v_4$ are linearly independent

basis of im(A) = $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ ($\vec v_1$, $\vec v_4$)

dim(im A) = 2

check:
$\dim(\ker A) = m - \dim(\operatorname{im} A)$
$= 5 - 2$
$= 3$ ✓

$\vec v_2 - 2\vec v_1 = \vec 0 \Rightarrow \vec w_2 = \begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}$

$1\vec v_3 = \vec 0 \Rightarrow \vec w_3 = \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}$

$-1\vec v_1 - 1\vec v_4 + 1\vec v_5 = \vec 0 \Rightarrow \vec w_5 = \begin{bmatrix}-1\\0\\0\\-1\\1\end{bmatrix}$

guess dim(ker A) = 3 ✓

basis of ker(A): $\vec w_2$, $\vec w_3$, $\vec w_5$

$$\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\-1\\1\end{bmatrix}$$

Check that $\vec w_5$ is in kernel:

$$-1\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + 0\begin{bmatrix}1\\ \vec v_2\\ 1\end{bmatrix} + 0\begin{bmatrix}1\\ \vec v_3\\ 1\end{bmatrix} - 1\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + 1\begin{bmatrix}2\\3\\4\\5\end{bmatrix} = \vec 0$$

$$\begin{bmatrix}-1 & & -1 & +2\\ -1 & & -2 & +3\\ -1 & & -3 & +4\\ -1 & & -4 & +5\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} ✓$$

$\vec w_5$ is in ker(A)

tying concepts together:

Find bases of kernel and image by inspection:

- separate redundant and non-redundant vectors
- express each redundant column vector as LC of preceding columns

  then write a corresponding linear relation to generate vector(s) of coefficients in ker(A)

  ↑ these vectors form basis of ker(A)

- non-redundant vectors form a basis of im(A)